# Note on an $E$-Plane Waveguide Step with Simultaneous Change of Media

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***Abstract*—A simple quasi-static formula for the discontinuity susceptance of a 2:1 *E*-plane waveguide step with a simultaneous change of media is found using the theory of singular integral equations. An important difference from previous solutions by this method comes from the determination of the constants in the solution.**

## I. Introduction

Many techniques are available to investigate the effects of an $E$-plane waveguide step discontinuity. Among these, the conformal mapping solution [1] is perhaps the most commonly used. Other techniques such as the modified residue calculus method [2] and direct mode matching are also available. When a simultaneous change of media (see Fig. 1) occurs at the step, the conformal mapping solution is no longer applicable. Although the other techniques mentioned can still be used, they do not display the simplicity that was characteristic of the conformal mapping solution of the empty waveguide step.

This short paper investigates the $E$-plane parallel-plate step discontinuity with a simultaneous change of media by applying the theory of singular integral equations [3]. In order to use this theory, the step ratio must be integrally related [4]; and to this end, a 2:1 step is analyzed. The solution obtained is simple, involving only Euler's psi-function, which is well tabulated [5]. An account of the difficulty in treating a nonintegrally related step ratio is given in [6], and the same difficulty persists with the present case of unequal media.

## II. Theory

In the region $z<0$, the transverse fields are

$$E_y(y, z) = e^{-jk_1 z} + Re^{jk_1 z} + \sum_{n=1}^{\infty} R_n e^{\Gamma_n z} \cos\frac{n\pi y}{b}$$

$$H_x(y, z) = -Y_1^0[e^{-jk_1 z} - Re^{jk_1 z}] + j\sum_{n=1}^{\infty} R_n Y_n e^{\Gamma_n z} \cos\frac{n\pi y}{b} \qquad (1)$$

where

$$Y_1^0 = \omega\epsilon_1/k_1 \quad k_1 = \omega\sqrt{\mu_1\epsilon_1} \quad Y_n = \omega\epsilon_1/\Gamma_n \quad \Gamma_n = \sqrt{\left(\frac{n\pi}{b}\right)^2 - k_1^2}.$$

Similarly, in the region $z>0$,

$$E_y(y, z) = T_0 e^{-jk_2 z} + \sum_{n=1}^{\infty} T_n e^{-\Gamma_n' z} \cos\frac{n\pi}{2b} y$$

$$H_x(y, z) = -Y_2^0 T_0 e^{-jk_2 z} - j\sum_{n=1}^{\infty} T_n Y_n' e^{-\Gamma_n' z} \cos\frac{n\pi}{2b} y \qquad (2)$$

where

$$Y_2^0 = \omega\epsilon_2/k_2 \quad k_2 = \omega\sqrt{\mu_2\epsilon_2} \quad Y_n' = \omega\epsilon_2/\Gamma_n' \quad \Gamma_n' = \sqrt{\left(\frac{n\pi}{2b}\right)^2 - k_2^2}.$$

Note that we are assumming that only the dominant TEM modes propagate on either side of the discontinuity.

In a manner similar to Lewin [4], one may match the fields at $z=0$ and arrive at the following quasi-static integral equation:

$$-Y_1^0(1-R) + \frac{Y_2^0}{2}(1+R) = -j\omega\epsilon_1\frac{2b}{\pi^2}\int_0^{\pi} E(\theta)\sum_{n=1}^{\infty}\frac{\cos n\theta \cos n\phi}{n}\, d\theta$$

$$-j\omega\epsilon_2\frac{2b}{\pi^2}\int_0^{\pi} E(\theta)\sum_{n=1}^{\infty}\frac{\cos n\theta/2 \cos n\phi/2}{n}\, d\theta, \qquad 0<\phi<\pi \qquad (3)$$

where the transformation of variables $\pi y/b=\theta$ has been used and $E(\theta)$ is the unknown electric field at $z=0$. In order to arrive at a singular integral equation, the derivative with respect to $\phi$ of (3) must first be taken. Then following closely the procedure used to solve a similar $H$-plane waveguide step [4], we may arrive at the following singular integral equation:

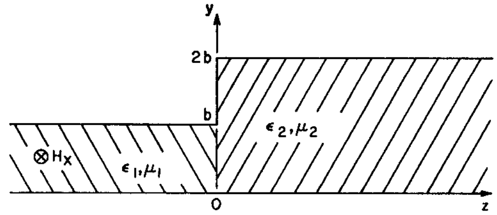


Fig. 1. Step discontinuity geometry.

$$P\int_{-\pi}^{\pi}\frac{E(\theta)}{\sin\theta/2 - \sin\phi/2}\left\{\cos\theta/2 + \left(1+\frac{2\epsilon_1}{\epsilon_2}\right)\cos\phi/2\right\} d\theta = 0$$

$$-\pi<\phi<\pi \qquad (4)$$

where the integral is a Cauchy principle value. Upon making the substitution

$$\alpha = \sqrt{1+\frac{2\epsilon_1}{\epsilon_2}}$$

and the change of variables

$$y = \sin\phi/2 \qquad x = \sin\theta/2$$

(4) becomes

$$P\int_{-1}^{1}\frac{F(x)}{x-y}\{\sqrt{1-x^2} + \alpha^2\sqrt{1-y^2}\}\, dx = 0, \qquad -1<y<1 \qquad (5)$$

where $F(x)dx = E(\theta)d\theta$. (No confusion should arise from the $x$, $y$ variables of (5) and the coordinate variables $x$, $y$.)

Equation (5) is of a similar form as the equation found for the asymetrical $H$-plane waveguide step [4], except that (5) is homogeneous. Equation (5) may be solved by reduction to Carleman form. Again, paralleling the previous development we find as a solution of (5)

$$F(x) = \frac{C_1}{2(x-C_2)}\left\{\frac{2\alpha^2(1+R)}{\pi(1+\alpha^2)}\left[\left(\frac{1-x}{1+x}\right)^{1-\beta} + \left(\frac{1-x}{1+x}\right)^{\beta}\right]\right.$$

$$\left. + \frac{\alpha}{\sqrt{1+\alpha^2}}\left(\frac{1-x}{1+x}\right)^{1-\beta}\frac{C_3}{1-x} + \frac{\alpha}{\sqrt{1+\alpha^2}}\left(\frac{1-x}{1+x}\right)^{\beta}\frac{C_4}{1-x}\right\} \qquad (6)$$

where

$$\beta = \frac{1}{\pi}\tan^{-1}_{(0,\pi)}\alpha$$

and $C_1$, $C_2$, $C_3$, and $C_4$ are undetermined constants. Now the form of the electric field is symmetrical about $y=0$, and this is sufficient to determine $C_2$, $C_3$, and $C_4$. Since $F(x)$ must be even we find the unique solution

$$C_2 = 1 \qquad C_3 = C_4 = 0.$$

The multiplier $C_1$ may be found by noting that

$$\int_{-1}^{1} F(x)\, dx = 2\pi(1+R).$$

Hence, we find

$$F(x) = \frac{\alpha(1+R)}{\sqrt{1+\alpha^2}}\left(\frac{X^{1-\beta} + X^{\beta}}{1-x}\right) \qquad (7)$$

where

$$X = \left(\frac{1-x}{1+x}\right).$$

The one remaining unknown, the reflection coefficient, is determined by recourse to the original integral equation (3). Since $R$ is a constant, independent of $\phi$, we may conveniently evaluate the integrals encountered when substituting (7) into (3) by letting $\phi=0$. Upon summing the series, one encounters integrals of the form

$$P\int_{-1}^{1}\frac{X^{1-\beta}+X^{\beta}}{1-x}\ln|x|\, dx$$

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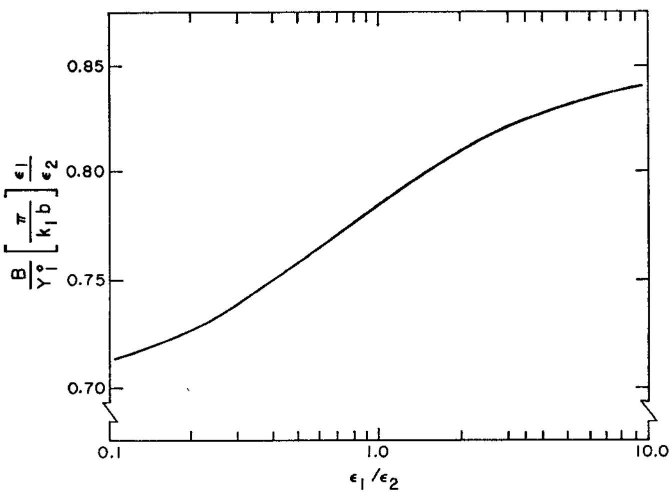


Fig. 2. Susceptance as a function of $\epsilon_1/\epsilon_2$.

and

$$P\int_{-1}^{1} \frac{X^{1-\beta}+X^{\beta}}{1-x}\ln\left|1-\sqrt{1-x^2}\right| dx.$$

These integrals may be evaluated by changing the integration variable to $X$, replacing the logarithmic terms in the integrand by appropriate integrals involving dummy variables, and then reversing orders of integration. After some lengthy manipulations, one finds the following expression for the input admittance at $z=0$:

$$\frac{Y_{\text{in}}}{Y_1^0} = \frac{1}{2}\frac{Y_2^0}{Y_1^0} + j\frac{B}{Y_1^0} \tag{8}$$

where

$$\frac{B}{Y_1^0} = \frac{k_1 b}{\pi}\alpha^2\frac{\epsilon_2}{\epsilon_1}\left[\frac{\pi}{2\alpha} - 2\ln 2 - \Psi(1-\beta) - \gamma\right] \tag{9}$$

where $\Psi(x)$ is Euler's psi-function and $\gamma$ is Euler's constant.

## III. Discussion of Results

In order to ascertain the validity of the solution, two convenient checks are available.

The first check is the edge condition. It is well known that the behavior of the field near a sharp corner plays an important role in the uniqueness of the solution. Using (7), one can show that near the corner at $z=0$, the electric field behaves as

$$E(\Delta) \sim 0(\Delta^{-(1-2\beta)}) \tag{10}$$

where $\Delta=(b-y)$. For equal media, $\beta=1/3$, and the field behaves as

$$E(\Delta) \sim 0(\Delta^{-1/3})$$

a well-known result. In general, we may use Mittra and Lee [2] and find that

$$E(\Delta) \sim 0(\Delta^{\tau-1}) \tag{11}$$

where

$$\tau - 1 = -\frac{1}{\pi}\tan^{-1}\left(\frac{1}{m}\sqrt{1+2m}\right)$$

and $m=\epsilon_1/\epsilon_2$. By using a trigonometric identity, it is seen that the results in (10) and (11) are identical.

Another check is a comparison with the known solution of the case $\epsilon_1=\epsilon_2$. Hence, $\beta=1/3$, and we have

$$\frac{B}{Y_1^0} = \frac{4b}{\lambda}\left[\frac{9}{4}\ln 3 - 3\ln 2\right]$$

which agrees with the predominant term of Marcuvitz [1].

Hence, (9) should be accurate to within a few percent for reasonable choices of material parameters.

Fig. 2 illustrates the behavior of $(B/Y_1^0)(\pi/k_1b)(\epsilon_1/\epsilon_2)$ as a function of $m=\epsilon_1/\epsilon_2$. It is seen to be nearly constant over at least two decades of $m$. It may also be noted that, for the quasi-static approximation used here, the permeabilities do not appear in the expression for the discontinuity susceptance, which is purely capacitive.

## IV. Conclusions

This short paper has found the effects of a 2:1 $E$-plane waveguide step with a simultaneous change of media. The solution has been found by application of the theory of singular integral equations. It should be noted that the conformal mapping solution is no longer valid because of the change in media. Other methods may be used to solve this problem for an arbitrary step ratio; however, these solutions do not exhibit the simplicity of (9). Perhaps in its simplicity, the solution obtained here can guide one in the effects of such a discontinuity. If desired, one might then proceed to a more complete solution.

## References

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# The Synthesis of Quarter-Wave Coupled Circulators with Chebyshev Characteristics

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***Abstract*—The purpose of this short paper is to give an exact theory of quarter-wave coupled circulators with Chebyshev characteristics. The synthesis starts by replacing the lumped-element equivalent shunt resonator of the circulator by a distributed one that has the same susceptance slope parameter as the original circuit. In this way the overall network involves commensurate transmission lines only. The bandwidth over which the assumed form of the equivalent circuit applies is carefully discussed in terms of the two split frequencies of the magnetized junction. Tables for the required circulator parameters and transformer admittances for one and two transformer sections as a function of VSWR and bandwidth are included. The realizable solution for the latter arrangement is severely restricted by the equivalent circuit of the basic junction. Experimental results on an octave-band stripline circulator, with a two-section transformer, are also included.**

## I. Introduction

An important property of the 3-port junction circulator is that an ideal circulator is obtained when the junction is matched [1]. One well-known method of broad-banding this device is to use external matching networks. One arrangement that is often used consists of a cascade of quarter-wave transformers [2]–[4].

An approximate theory for this type of network has been given in [2]. The purpose of this short paper is to give an exact synthesis procedure for the case of one- and two-step transformers that will give an equal-ripple Chebyshev response for the reflection coefficient of the overall circulator network. This short paper assumes in the usual way that the equivalent network at the reference terminals of the junction consists of a shunt lumped-element resonator in parallel with the gyrator conductance of the circulator [4]–[7]. The synthesis procedure then starts by replacing the lumped-element resonator by a distributed one consisting of a quarter-wave short-circuited transmission line that has the same susceptance slope parameter as the original circuit. The two circuits are equivalent provided their susceptance slope parameters are the same. The equivalent circuit of the device can therefore be represented by a quarter-wave short-circuited transmission line in parallel with the gyrator conductance of the circulator. The admittance of the distributed network is uniquely related to the susceptance slope parameter, once the nature of the net-

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